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LETTER TO THE EDITOR

Anisotropic spiral self-avoiding walks

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Abstract. Manna has recently studied two extensions of spiral self-avoiding walks on the square lattice, with spiral constraints after East and West steps, but not after North and South steps. Here we calculate the connective constants for these two models.

In a recent paper, Manna (1984) has proposed two new self-avoiding walk models which incorporate partial spiral constraints (see Joyce (1984) and references therein). Both models involve self-avoiding walks on the square lattice and we label the four possible steps on this lattice N, E, S, W in an obvious notation. However, both models have additional constraints. In the first (which we shall call two choice anisotropic spirals or model A), if the nth step is N or S, the $(n+1)$th step cannot be either N or S, if the nth step is E the $(n+1)$th step cannot be S, and if the nth step is W the $(n+1)$th step cannot be N. In the second (three choice anisotropic spirals or model B), the walks are self-avoiding and, in addition, a W(E) step cannot be followed by an N(S) step. (Model B is less restrictive than model A in that an N step can be followed by an N step and an S step can be followed by an S step.)

Manna (1984) studied these models using exact enumeration and series analysis techniques. He *assumed* that the number, $s_n(\mathrm{A})$, of n-step self-avoiding walks subject to model A restrictions could be written as

$$s_n(\mathrm{A}) \sim n^{g(\mathrm{A})}\mu(\mathrm{A})^n \tag{1}$$

with a similar expression for model B. A necessary condition for (1) is that

$$0 < \lim_{n\to\infty} n^{-1} \log s_n(\mathrm{A}) = \log \mu(\mathrm{A}) < \infty. \tag{2}$$

Although (2) is valid for many self-avoiding walk models, it is known that the number (s_n) of *spiral* self-avoiding walks behaves asymptotically as

$$s_n \sim \rho^{n^{1/2}} \tag{3}$$

so that, although the limit corresponding to (2) exists, its value is zero (see e.g. Joyce 1984).

In this letter we show that (2) holds both for model A and for model B, and we calculate $\mu(\mathrm{A})$ and $\mu(\mathrm{B})$ exactly.

We first note that standard concatenation arguments establish the existence of the limits (Hammersley and Morton 1954). The set of graphs obtained by joining end-to-end an n-step model A walk and an m-step model A walk will include all $(n+m)$-step model A walks, so that

$$s_{n+m}(\mathrm{A}) \geq s_m(\mathrm{A})s_n(\mathrm{A}) \tag{4}$$

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This, together with the observation that $m^{-1}\log s_m(\mathrm{A})$ is bounded below by zero is sufficient to establish that

$$\lim_{n\to\infty} n^{-1}\log s_n(\mathrm{A}) = \inf_{n>0} n^{-1}\log s_n(\mathrm{A}) = \log\mu(\mathrm{A}). \tag{5}$$

Clearly the limit is not infinite, because $s_m(\mathrm{A}) < 4^m$, but to show that the limit is not zero we need a lower bound on $s_m(\mathrm{A})$.

We study the subset of model A walks in which W and S steps are forbidden. If b_n is the number of these walks with n steps, then b_n satisfies the relation

$$b_n = b_{n-1} + b_{n-2} \tag{6}$$

since if an n-step walk begins with an E step it can be completed in b_{n-1} ways, while if it begins with an N step, the next step must be E and the remaining $(n-2)$ steps of the walk can be completed in b_{n-2} ways. It follows immediately that

$$\lim_{n\to\infty} n^{-1}\log b_n = \log[(1+\sqrt{5})/2]. \tag{7}$$

We can derive a similar lower bound on $\mu(\mathrm{B})$ by considering the subset of model B walks in which every step is either N or E. Clearly such walks are self-avoiding and have two possibilities available at each step, so that

$$\mu(\mathrm{B}) \geq 2. \tag{8}$$

(It is not difficult to show that forbidding only W steps in model B gives the same result.)

We now argue that

$$\mu(\mathrm{A}) = (1+\sqrt{5})/2 \tag{9}$$

and that

$$\mu(\mathrm{B}) = 2. \tag{10}$$

The argument is somewhat simpler for model B and we therefore consider that case in detail. Each model B walk can be written as

$$\mathrm{E}^{p_{11}}\mathrm{N}^{p_{12}}\mathrm{E}^{p_{13}}\ldots\mathrm{N}^{p_{1a}}\mathrm{W}^{q_{11}}\mathrm{S}^{q_{12}}\mathrm{W}^{q_{13}}\ldots\mathrm{S}^{q_{1b}}\mathrm{E}^{p_{21}}\mathrm{N}^{p_{22}}\ldots$$

i.e. as a sequence of north-east steps, followed by a sequence of south-west steps, followed by a sequence of north-east steps, etc. If we write

$$\sum_j p_{kj} = p_k \tag{11}$$

and

$$\sum_j q_{kj} = q_k \tag{12}$$

we see that the number of ways of taking the first p_1 steps is at most 2^{p_1}, the number of ways of taking the next q_1 steps is at most 2^{q_1}, and so on, so that

$$s_n(\mathrm{B}) \leq \sum{}' 2^{p_1}2^{q_1}2^{p_2}2^{q_2}\ldots = 2^n \sum{}' 1 \tag{13}$$

where Σ' is a sum over p_k, q_k subject to certain restrictions. Clearly

$$p_1 + q_1 + p_2 + q_2 + \ldots = n. \tag{14}$$

In addition, because of the spiral constraint imposed after W or E steps, $p_1 < q_1 < p_2 < q_2 \dots$ if the walk is to be indefinitely extendible. If not, this set of inequalities will be violated at some stage and the inequalities will then be reversed for the duration of the walk. In either case, there are only $\exp(O(\sqrt{n}))$ such partitions of n, and

$$2^n \leq s_n(\mathrm{B}) \leq 2^n \exp(O(\sqrt{n})) \tag{15}$$

so that

$$\mu(\mathrm{B}) = 2. \tag{16}$$

By a similar argument we can show that

$$\mu(\mathrm{A}) = (1+\sqrt{5})/2 \tag{17}$$

and we note that the numerical estimates (Manna 1984) of $\mu(\mathrm{A})$ and $\mu(\mathrm{B})$ are rather close to these values.

These results are not strong enough either to confirm or rule out the form (1) assumed by Manna (1984) but they do establish that the numbers of such walks grow exponentially, as do the numbers of ordinary self-avoiding walks (Hammersley and Morton 1954), and not like the numbers of pure spiral walks (see for instance Joyce 1984).

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